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ORIGINAL ARTICLE

Bernstein method for the MHD flow and heat transfer of a second grade fluid in a channel with porous wall

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Abstract In this paper, we present an approximate solution method for the problem of magnetohydrodynamic (MHD) flow and heat transfer of a second grade fluid in a channel with a porous wall. The method is based on the Bernstein polynomials with their operational matrices and collocation method. Under some regularity conditions, upper bounds of the absolute errors are given. We apply the residual correction procedure which may estimate the absolute error to the problem. We may estimate the absolute error by using a procedure depends on the sequence of the approximate solutions. For some certain cases, we apply the method to the problem in the numerical examples. Moreover, we test the impact of changing the flow parameters numerically. The results are consistent with the results of Runge-Kutta fourth order method and homotopy analysis method.
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1. Introduction

Magnetohydrodynamic (MHD) studies consider the dynamics of electrically conducting fluids [1]. MHD consist of a set of equations, which are a combination of the Navier–Stokes equations of fluid dynamics and Maxwells equations of electromagnetism [1]. MHD equations appear in many areas of physics and engineering, such as liquid-metal cooling of nuclear reactors [2,3], electromagnetic casting of metals [4], controlling thermonuclear fusion and plasma confinement [5,6], climate change forecasting, and sea water propulsion [7,8].

Recently, Tezer-Sezgin [9] used the polynomial based differential quadrature and the Fourier expansion based differential quadrature method to solve MHD flow equations in a rectangular duct in the presence of a transverse external oblique magnetic field. As analytic solution of MHD, homotopy analysis method and homotopy perturbation method or some of their modifications were used under the different regions

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and conditions [10–22]. As a numerical solution of MHD, finite difference and finite element method was considered in several studies [23–29] and given a comprehensive stability and error analysis. Guermond et al. [30] used Lagrange finite elements and an Interior Penalty technique to solve MHD. They gave the method with stability and convergence analysis. Shakeri and Dehghan [31] solved MHD flow through a pipe of rectangular section by using combined finite volume method and spectral element technique. As a collocation method, Çelik used Chebyshev collocation method [32] to solve MHD flow equations in a rectangular duct in the presence of transverse external oblique magnetic field.

In this study, we consider the problem introduced by Parida et al. [33]. They studied the laminar flow of a second grade electrically-conducting viscoelastic fluid between two parallel plates and introduced the following nonlinear system:

$$Re(f'f''-ff''')-f^{iv}-K\,Re(f'f^{iv}-ff^{v})+M^2f''=0, \tag{1}$$

$$\theta''+Pr\,Ref\theta'=0, \tag{2}$$

subject to the boundary conditions

$$\begin{cases} f(\eta)=1, \quad f'(\eta)=0, \quad \text{at} \quad \eta=1, \\ f(\eta)=0, \quad f'(\eta)=0, \quad \text{at} \quad \eta=0, \\ \theta(0)=1, \quad \theta(1)=0. \end{cases} \tag{3}$$

Here, Re is the Reynolds number, M is the dimensionless Hartmann number, K is the non-Newtonian character of the fluid and Pr is the Prandtl number. The physical meanings of these numbers can be found in [33]. Parida et al. [33] first reduced (1) and (2) to system of first-order differential equations, with the help of the boundary conditions, which were then solved by the fourth-order Runge-Kutta method. An approximate analytical solution of (1)–(3) based on the homotopy analysis method was presented by Raftari et al. [22].

Yousefi and Behroozifar [34] derived the operational matrices of derivative and integral of Bernstein polynomials. They applied these matrices forms to solve Bessel differential equation and EmdenFowler equation numerically. Rad et al. [35] constructed a method, Bernstein integral operational matrix to fractional calculus, based on using the operational matrix of B-polynomials. They applied the method to solve linear and nonlinear fractional differential equations. Recently, Bernstein operational matrix method was used to solve high order linear Volterra–Fredholm integro-differential equations [36], Riccati equation and Volterra population model [37], parabolic partial differential equation with boundary integral conditions [38], the nonlinear age-structured population models [39], and nonlinear Volterra–Fredholm–Hammerstein integral equations [40].

In this paper, we propose an approximate solution method based on the Bernstein polynomials with their operational matrices and collocation method to solve the problems (1)–(3) numerically. To obtain the approximate solutions of the problem, we first approximate f and θ as

$$f(\eta)\simeq f_m(\eta)=\sum_{i=0}^{m}c_{1,i}\mathrm{B}_{i,m}(\eta) \tag{4}$$

$$\theta(\eta)\simeq \theta_m(\eta)=\sum_{i=0}^{m}c_{2,i}\mathrm{B}_{i,m}(\eta) \tag{5}$$

where $c_{1,i}$ and $c_{2,i}$ are unknown coefficients for $0\leqslant i\leqslant m$.

The paper is organized as follows. In Section 2, we give some necessary definitions and theorems. We will use these theorems to bound the absolute errors. In Section 3, the fundamental matrix relations for the method are presented. The problem is converted to an algebraic nonlinear system. For error analysis, upper bounds for the absolute errors are obtained by using the error formula of polynomial interpolation. Two procedures to estimate the error are applied to the problem in Section 4. Numerical examples are given in Section 5. The numerical results are coincided with the results of Runge-Kutta fourth order method [33] and homotopy analysis method [22].

2. Bernstein polynomials and their operational matrices

Bernstein polynomials of degree m are defined by [41]

$$\mathrm{B}_{i,m}(\eta)=\binom{m}{i}\eta^i(1-\eta)^{m-i}, \quad i=0,1,\ldots,m, \quad \eta\in[0,1],$$

where the binomial coefficient is

$$\binom{m}{i}=\frac{m!}{i!\,(m-i)!}.$$

For convenience, we usually set $\mathrm{B}_{i,m}=0$ if $i<0$ or $i>m$. Bernstein polynomials are used to solve differential equations with collocation method [42–45].

For given $n+1$ pairs (x_i,y_i), the interpolating polynomial p_m is the polynomial that interpolates on the set (x_i,y_i), $i=0,1,\ldots,n$, i.e.,

$$p_m(x_i)=c_0+c_1x_i+\cdots+c_mx_i^m=y_i, \quad i=0,1,\ldots,n.$$

Here, $x_i, i=0,1,\ldots,n$ are called the interpolation nodes.

Theorem 1 [46]. *Given $n+1$ distinct nodes $x_0,x_1,\ldots,x_n$ and $n+1$ corresponding values $y_0,y_1,\ldots,y_n$, then there exist a unique polynomial $p_n\in P_n$ such that $p_n(x_i)=y_i$ for $i=0,1,\ldots,n$.*

Theorem 2 [46]. *Let $x_0,x_1,\ldots,x_n$ be $n+1$ distinct nodes and let x be a point belonging to the domain of a given function f. Assume that $f\in C^{n+1}(I_x)$, where I_x is the smallest interval containing the nodes $x_0,x_1,\ldots,x_n$ and x. Then the interpolation error at the point x is given by*

$$f(x)-p_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$$

where $\xi\in I_x$.

3. Bernstein operational matrix method

Let us find the fundamental matrix relations f and θ with their derivatives. First, let us write f and θ as matrix forms

$$f(\eta)\simeq f_m(\eta)=\sum_{i=0}^{m}c_{1,i}\mathrm{B}_{i,m}(\eta)=\mathrm{C}_1^{\mathrm{T}}\Phi(\eta), \tag{6}$$

$$\theta(\eta)\simeq \theta_m(\eta)=\sum_{i=0}^{m}c_{2,i}\mathrm{B}_{i,m}(\eta)=\mathrm{C}_2^{\mathrm{T}}\Phi(\eta), \tag{7}$$

where

$C_j^T = [c_{j,0}, c_{j,1}, \ldots, c_{j,m}],$

$\Phi(\eta) = [B_{0,m}(\eta), B_{1,m}(\eta), \ldots, B_{m,m}(\eta)]^T.$

By using the binomial expansion of $(1-x)^{m-i}$, we get

$$\begin{aligned}B_{i,m}(\eta) &= \binom{m}{i} x^i (1-x)^{m-i}\\ &= \binom{m}{i} x^i \left(\sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} x^k\right)\\ &= \sum_{k=0}^{m-i} (-1)^k \binom{m}{i}\binom{m-i}{k} x^{k+i}, \quad i = 0,1,\ldots,m.\end{aligned}$$

Let A_{i+1} be a $1 \times (m+1)$ matrix and be defined as follows:

$$(A_{i+1})_{1,j} = \begin{cases} 0, & j \leqslant i \\ (-1)^{j-i}\binom{m}{i}\binom{m-i}{j-i}, & j > i \end{cases}$$

Then $B_{i,m}(\eta)$ can be written as follows:

$$B_{i,m}(\eta) = A_{i+1} T_m(\eta)$$

where

$$T_m(\eta) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^m \end{bmatrix}.$$

Therefore, we can write $\Phi(\eta)$ as

$$\Phi(\eta) = A T_m(\eta)$$

where

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_m \end{bmatrix}.$$

Let us find the matrix representation for $\Phi'(\eta)$.

$$\begin{aligned}\frac{d}{dx}\Phi(\eta) &= A \frac{d}{dx} T_m(\eta)\\ &= A \begin{bmatrix} 0 \\ 1 \\ 2\eta \\ \vdots \\ m\eta^{m-1} \end{bmatrix}\\ &= A \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m \end{bmatrix} \begin{bmatrix} 1 \\ \eta \\ \eta^2 \\ \vdots \\ \eta^m \end{bmatrix}\\ &= A\Lambda T_m(\eta)\end{aligned}$$

where

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m \end{bmatrix}.$$

Expanding $T_m(\eta)$ in terms of Bernstein polynomial bases yields

$T_m(\eta) = A^{-1}\Phi(\eta).$

Thus

$f(\eta) = D\Phi(\eta)$

where D is the operational matrix of derivative

$D = A\Lambda A^{-1}.$

To get the matrix forms of $f^{(n)}$, we first find the matrix representation of $\Phi^{(n)}(\eta)$. By using the following procedure

$$\Phi''(\eta) = (\Phi'(\eta))' D\Phi'(\eta) = D^2\Phi(\eta)$$

$$\vdots$$

$$\Phi^{(n)}(\eta) = D(\Phi^{(n-1)}(\eta)) = \ldots = D^n\Phi(\eta),$$

we obtain the following:

$$f^{(n)}(\eta) = C_1^T D^n \Phi(\eta). \tag{8}$$

We can find the matrix representation of $\theta^{(n)}$ by a similar way

$$\theta^{(n)}(\eta) = C_2^T D^n \Phi(\eta). \tag{9}$$

Substituting (6)–(9) into (1) and (2) yields the residuals $R_1(\eta)$ and $R_2(\eta)$ for Eqs. (1) and (2) as Substituting (6)–(8) into (1) and (2) yields the residuals $R_1(\eta)$ and $R_2(\eta)$ for Eqs. (1) and (2) as

$$\begin{aligned}R_1(\eta) = {} & Re\left(C_1^T D\Phi C_1^T D^2\Phi - C_1^T\Phi C_1^T D^3\Phi\right) - C_1^T D^4\Phi\\ & - KRe\left(C_1^T D\Phi C_1^T D^4\Phi - C_1^T\Phi C_1^T D^5\Phi\right)\\ & + M^2 C_1^T D^2\Phi,\end{aligned} \tag{10}$$

$$R_2(\eta) = C_2^T D^2\Phi + Pr\,Re C_1^T\Phi C_2^T D\Phi. \tag{11}$$

To find the unknown coefficients $c_{i,j}, i = 1,2, j = 0,1,\ldots,m$, we need $2m+2$ algebraic equations. Let $\eta_i, i = 0,1,\ldots,m-2$ be the collocation nodes. First, putting the nodes $\eta_i, i = 0,1,\ldots,m-4$ into (10) gives $m-3$ algebraic equations. Similarly, we get $m-1$ algebraic equations by applying the nodes to (11). We select the nodes as the roots of Chebyshev polynomials, as a special case,

$$\eta_i = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{(2i+1)\pi}{2m}\right), \quad i = 0,1,\ldots,m-2. \tag{12}$$

Second, imposing the boundary condition (3) into Eqs. (6)–(8) gives 6 more linear algebraic equations:

$$\begin{cases} f(1) = C_1^T\Phi(1) = 1, & f'(1) = C_1^T\Phi(1) = 0,\\ f(0) = C_1^T\Phi(0) = 0, & f'(0) = C_1^T\Phi(0) = 0,\\ \theta(0) = C_2^T\Phi(0) = 1, & \theta(1) = C_2^T\Phi(1) = 0, \end{cases} \tag{13}$$

Thus, we have $2m+2$ set of linear or nonlinear algebraic equations for (1) and (2). Solving these equations gives the unknown coefficients of the vector $C_i^T, i = 1,2$. Consequently, f and θ given in Eqs. (6) and (7) are obtained.

4. Error analysis based on residual correction procedure

In this section, we first give the upper bounds for the absolute errors. Then, we constitute two different error estimation procedures which involve residual correction and Runge-Kutta-Fehlberg Method (RK45).

Let f, θ and f_m, θ_m be the exact solutions and the approximate solutions of (1) and (2), respectively. Now, let us write the interpolation polynomials of f and θ on the collocation nodes as

$$p_m = \sum_{i=0}^{m} c^*_{1,i} \mathrm{B}_{i,m}(\eta) = \mathrm{C}_1^{*\mathrm{T}} \Phi(\eta), \tag{14}$$

$$q_m = \sum_{i=0}^{m} c^*_{2,i} \mathrm{B}_{i,m}(\eta) = \mathrm{C}_2^{*\mathrm{T}} \Phi(\eta). \tag{15}$$

Then, the following upper bounds are obtained for the absolute errors.

Theorem 3. *With the same notation above, if $f, \theta \in C^{m+1}[0,1]$, then the absolute error is bounded as follows:*

$$|f - f_m| \leqslant \frac{f^{(m+1)}(\xi)}{(m+1)!}(\eta - \eta_0)\cdots(\eta - \eta_n) + \left|(\mathrm{C}_1^* - \mathrm{C}_1)^{\mathrm{T}} \Phi(\eta)\right|, \tag{16}$$

$$|\theta - \theta_m| \leqslant \frac{\theta^{(m+1)}(\zeta)}{(m+1)!}(\eta - \eta_0)\cdots(\eta - \eta_n) + \left|(\mathrm{C}_2^* - \mathrm{C}_2)^{\mathrm{T}} \Phi(\eta)\right|, \tag{17}$$

where $\xi, \zeta \in (0, 1)$.

Proof. Adding and subtracting the interpolation polynomials p_m and q_m with triangle inequality yields

$$|f - f_m| \leqslant |f - p_m| + |p_m - f_m|, \tag{18}$$

$$|\theta - \theta_m| \leqslant |\theta - q_m| + |q_m - \theta_m|. \tag{19}$$

Let us consider the first terms on the right hand sides. Applying Theorem 2 gives the following upper bounds:

$$|f - p_m| = \frac{f^{(m+1)}(\xi)}{(m+1)!}(\eta - \eta_0)\cdots(\eta - \eta_n), \tag{20}$$

$$|\theta - q_m| = \frac{\theta^{(m+1)}(\zeta)}{(m+1)!}(\eta - \eta_0)\cdots(\eta - \eta_n), \tag{21}$$

where $\xi, \zeta \in (0, 1)$. Using the fundamental matrix relations gives the following:

$$|p_m - f_m| = \left|\mathrm{C}_1^{*\mathrm{T}}\Phi(\eta) - \mathrm{C}_1^{\mathrm{T}}\Phi(\eta)\right| = \left|(\mathrm{C}_1^* - \mathrm{C}_1)^{\mathrm{T}}\Phi(\eta)\right|, \tag{22}$$

$$|q_m - \theta_m| = \left|\mathrm{C}_2^{*\mathrm{T}}\Phi(\eta) - \mathrm{C}_2^{\mathrm{T}}\Phi(\eta)\right| = \left|(\mathrm{C}_2^* - \mathrm{C}_2)^{\mathrm{T}}\Phi(\eta)\right|. \tag{23}$$

Summing of (20) and (22) and also (21) and (23) completes the proof. □

Now, let us constitute a residual correction procedure for the method. This procedure is based on the residual function. First, adding and subtracting the terms

$$Re\left(f'_m f''_m - f_m f'''_m\right) - f^{iv}_m - K\,Re\left(f'_m f^{iv}_m - f_m f^{v}_m\right) + M^2 f''_m = 0,$$
$$\theta''_m + Pr\,Re f_m \theta'_m = 0,$$

to (1) and (2), respectively, then attaining $e_{1,m} := f - f_m$ and $e_{2,m} := \theta - \theta_m$ yields

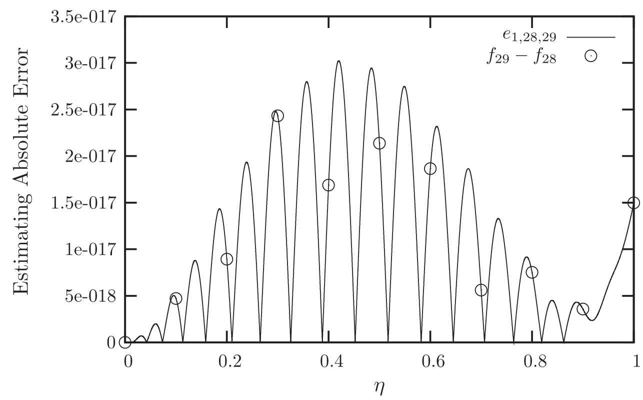

Figure 1 The estimation of absolute error to $f(\eta)$ for $m = 28, 29$ in the case of $Re = 1, K = 0.02, Pr = 1$, and $M = 6$.

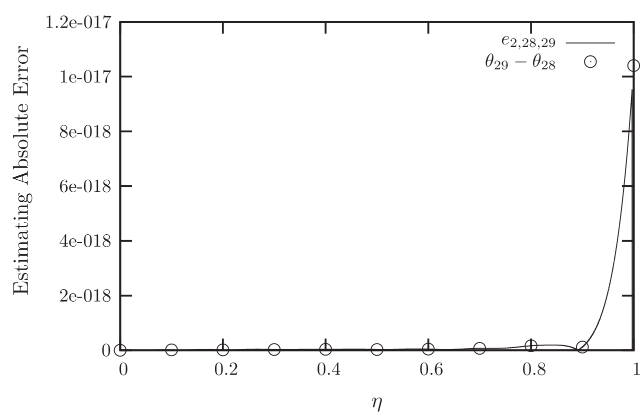

Figure 2 The estimation of absolute error to $\theta(\eta)$ for $m = 28, 29$ in the case of $Re = 1, K = 0.02, Pr = 1$, and $M = 6$.

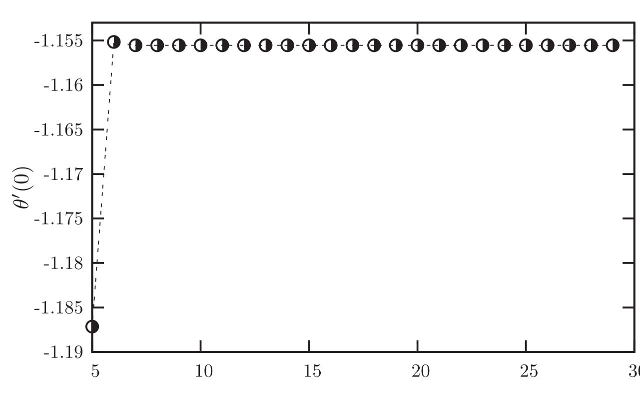

Figure 3 Convergence of Bernstein solutions for $\theta'(0)$ for different values of m in the case of $Re = 1, K = 0.02, Pr = 1$, and $M = 6$.

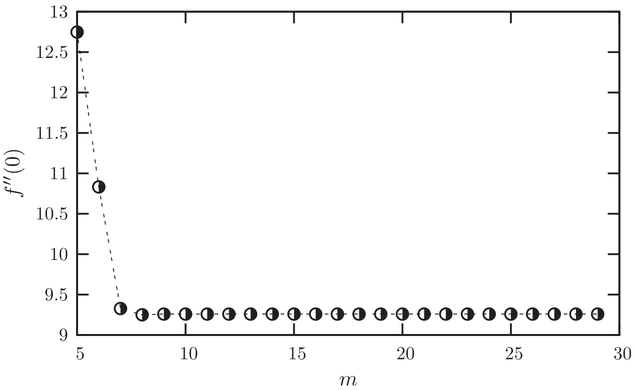

Figure 4 Convergence of Bernstein solutions for $f''(0)$ for different values of m in the case of $Re = 1, K = 0.02, Pr = 1$, and $M = 6$.

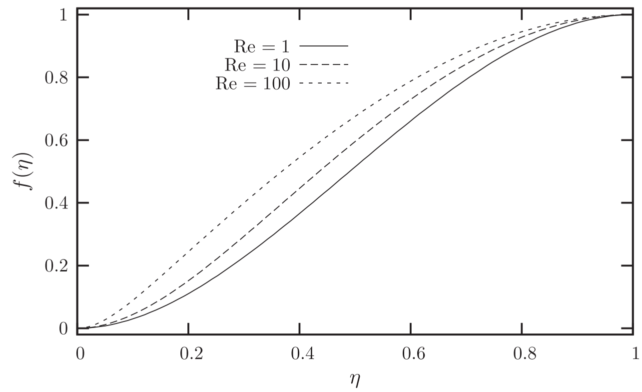

Figure 7 Graphs of normal velocity at different Reynolds parameters Re when $K = 0$ and $M = 1$.

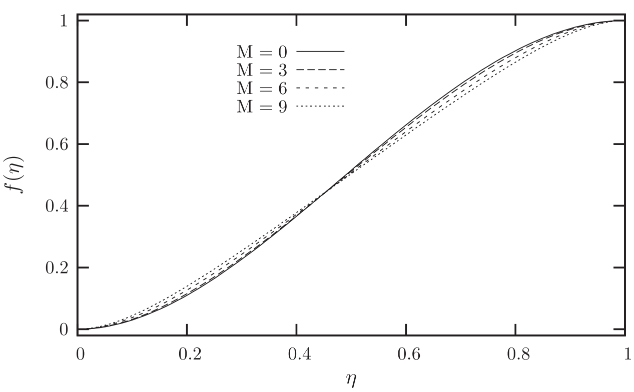

Figure 5 Graphs of normal velocity at different Hartmann parameters M when $K = 0$ and $Re = 2$.

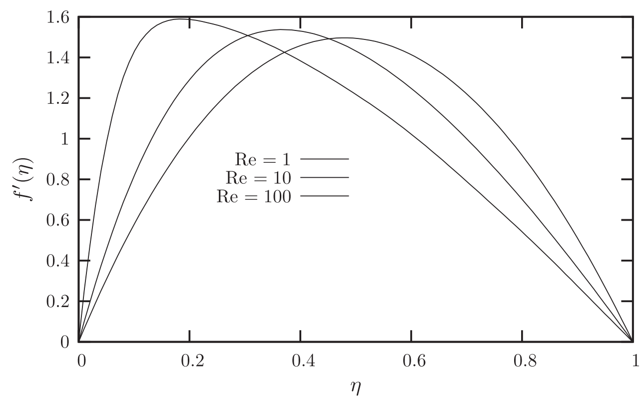

Figure 8 Graphs of tangential velocity at different Reynolds parameters Re when $K = 0$ and $M = 1$.

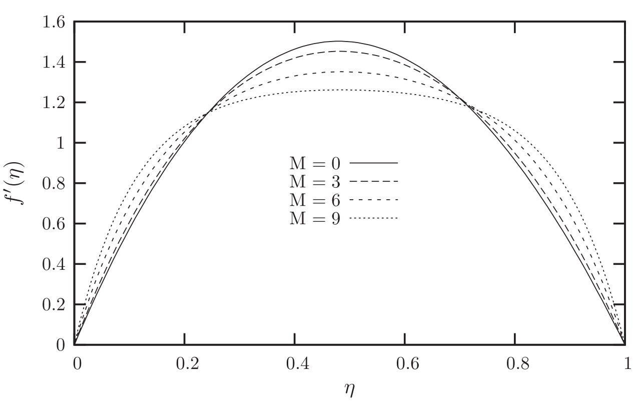

Figure 6 Graphs of tangential velocity at different Hartmann parameters M when $K = 0$ and $Re = 2$.

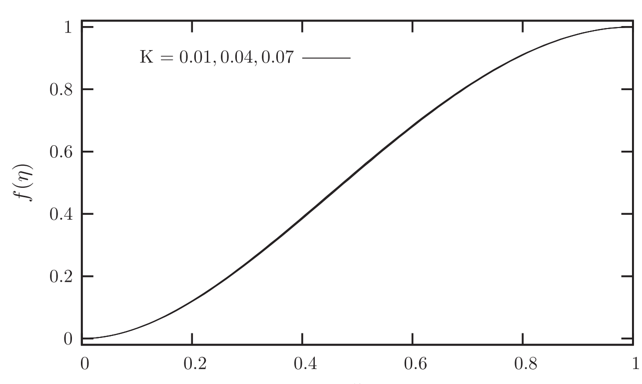

Figure 9 Graphs of normal velocity at different viscoelastic parameters K when $M = 0$ and $Re = 3$.

$$
\begin{aligned}
&Re\Big(e'_{1,m}e''_{1,m} + e'_{1,m}f''_m + e''_{1,m}f'_m - e_{1,m}e'''_{1,m} - e_{1,m}f'''_m - e'''_{1,m}f_m\Big) - e^{iv}_{1,m} \\
&\quad - K\,Re\Big(e'_{1,m}e^{iv}_{1,m} + e'_{1,m}f^{iv}_m + e^{iv}_{1,m}f'_m - e_{1,m}e^{v}_{1,m} - e_{1,m}f^{v}_m - e^{v}_{1,m}f_m\Big) \\
&\quad + M^2 e''_{1,m} + \mathrm{R}_1 = 0,
\end{aligned} \tag{24}
$$

$$
e''_{2,m} + Pr\,Re\Big(e_{1,m}e'_{2,m} + e_{1,m}\theta'_m - e'_{2,m}f_m\Big) + \mathrm{R}_2 = 0, \tag{25}
$$

where

$$
\begin{aligned}
&e_{i,m}(x) = \mathrm{C}_i^{e_{i,m},\mathrm{T}}\Phi(x), \quad i = 1,2 \\
&\mathrm{C}_i^{e_{i,m},\mathrm{T}} = [\mathrm{c}_{i,0}^{e_{i,m}}, \mathrm{c}_{i,1}^{e_{i,m}}, \ldots, \mathrm{c}_{i,m}^{e_{i,m}}], \quad i = 1,2 \\
&\mathrm{R}_1 = -Re\big(f'_m f''_m - f_m f'''_m\big) + f^{iv}_m + K\,Re\big(f'_m f^{iv}_m - f_m f^{v}_m\big) - M^2 f''_m, \\
&\mathrm{R}_2 = -\theta''_m - Pr\,Re f_m \theta'_m.
\end{aligned}
$$

Therefore, applying the method to the Eqs. (24) and (25) subject to the boundary conditions

$$\begin{cases} e_{1,m}(1) = C_1^{e_{1,m},T}\Phi(1) = 0, & e'_{1,m}(1) = C_1^{e_{1,m},T}\Phi(1) = 0, \\ e_{1,m}(0) = C_1^{e_{1,m},T}\Phi(0) = 0, & e'_{1,m}(0) = C_1^{e_{1,m},T}\Phi(0) = 0, \\ e_{2,m}(0) = C_2^{e_{2,m},T}\Phi(0) = 0, & e_{2,m}(1) = C_2^{e_{2,m},T}\Phi(1) = 0. \end{cases} \tag{26}$$

gives as the approximate solutions of the absolute errors, namely $e_{1,m,n}(\eta)$ and $e_{2,m,n}(\eta)$.

Corollary 1. *If f_m and θ_m are the approximate solutions of (1) and (2) respectively, then $f_m + e_{1,m,n}$ and $\theta_m + e_{2,m,n}$ are also approximate solutions for (1) and (2) respectively.*

Thus, the approximate solutions $f_m + e_{1,m,n}$ and $\theta_m + e_{2,m,n}$ are better approximations than f_m and θ_m if

$$\|e_{1,m} - e_{1,m,n}\| \leqslant \|f - f_m\|,$$
$$\|e_{2,m} - e_{2,m,n}\| \leqslant \|\theta - \theta_m\|.$$

We will call the approximate solutions $f_m + e_{1,m,n}$ and $\theta_m + e_{2,m,n}$ as the corrected approximate solutions.

As a second error estimation of the absolute error, we can use any two elements of the sequence of approximations. This estimation is similar to the error analysis of RK45. In the exact arithmetic, the absolute errors can be bounded by

$$|\|f - f_m\| - \|f - f_n\|| = C\|f - f_k\| \leqslant \|f_n - f_m\|,$$
$$|\|\theta - \theta_m\| - \|\theta - \theta_n\|| = C\|\theta - \theta_k\| \leqslant \|\theta_n - \theta_m\|, \quad m < n,$$

where

$$\|f - f_k\| = \max\{\|f - f_m\|, \|f - f_n\|\},$$
$$\|\theta - \theta_k\| = \max\{\|\theta - \theta_m\|, \|\theta - \theta_n\|\}.$$

Thus, if the error sequence is monotone, then we can use $\|f_{m+1} - f_m\|$ or $\|f_{m-1} - f_m\|$ to estimate the absolute error $\|f - f_m\|$. A similar conclusion can be said for $\|\theta - \theta_m\|$.

5. Numerical experiments

In this section we shall present some computed solutions for certain cases. First, we apply the method to (1) and (2) for $m = 9, Re = 1, K = 0.02, Pr = 1$, and $M = 6$. The velocity and temperature fields are obtained as

$$\begin{cases} f_9(\eta) = 4.62\eta^2 - 9.43\eta^3 + 13.86\eta^4 - 15.96\eta^5 + 13.93\eta^6 \\ \qquad -8.85\eta^7 + 3.49\eta^8 - 0.69\eta^9, \\ \theta_9(\eta) = 1 - 1.15\eta - 0.22 \times 10^{-4}\eta^2 + 0.99 \times 10^{-3}\eta^3 \\ \qquad +0.43\eta^4 - 0.48\eta^5 + 0.33\eta^6 - 0.19\eta^7 \\ \qquad +0.07\eta^8 - 0.1410714\eta^9, \end{cases} \tag{27}$$

respectively. We plot the estimation of absolute error obtained by residual correction procedure and the procedure obtained by consecutive approximations for $m = 28, 29$ and $Re = 1, K = 0.02, Pr = 1, M = 6$ in Figs. 1 and 2. Both estimations are consistent with each other. In Figs. 3 and 4, the values of $\theta'(0)$ and $f''(0)$ obtained by the method are drawn for $f''(0)$ and $\theta'(0)$ for different values of m and $Re = 1, K = 0.02, Pr = 1, M = 6$. The results show that the solutions converge to the point around -1.155 for $\theta'(0)$ and to the point between 9 and 9.5 for $f''(0)$ provided that m increases.

Next, we test numerically the impact of changing the flow parameters. The results are plotted in Figs. 5–10 for the velocity fields and Fig. 11 for the temperature field. In Figs. 5 and 6, the normal velocity and tangential velocity are plotted at different values of Hartmann number M when $K = 0$ and $Re = 2$. Normal velocity and tangential velocity at different values of Reynolds number Re when $K = 0$ and $M = 1$ are shown in Figs. 7 and 8. Normal velocity and tangential velocity at different values of the viscoelastic parameter K when $M = 0$ and $Re = 3$ are depicted in Figs. 9 and 10. Fig. 11 shows the graphs of temperature field at different values of Reynolds number Re when $M = 0$ and $Pr = 1$. All results obtained by the method agree well with the results of [33,22].

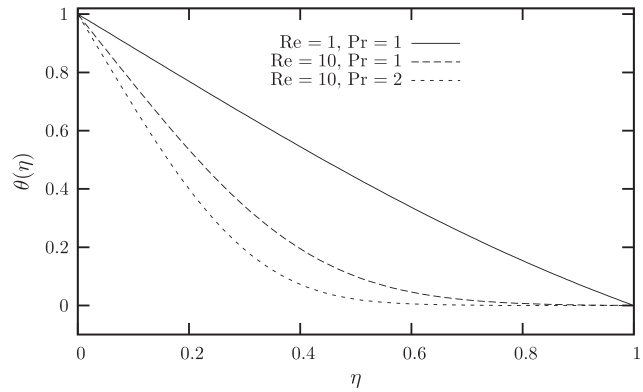

Figure 11 Graphs of temperature at different Reynolds Re when $M = 0$ and $Pr = 1$.

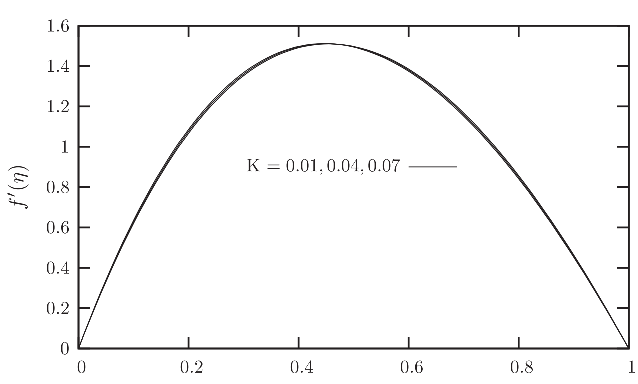

Figure 10 Graphs of tangential velocity at different viscoelastic parameters K when $M = 0$ and $Re = 3$.

6. Conclusions

In this study, we present an approximate solution method based on Bernstein polynomials with their operational matrices and collocation method to solve the system describing the laminar flow of a second grade electrically-conducting viscoelastic fluid between two parallel plates. Upper bound for the absolute errors was given. In the case of unknown exact solutions, two different methods which estimate the absolute errors were proposed. The numerical results are consistent with the results of Runge-Kutta fourth order method [33] and homotopy analysis method [22].

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